On a Family of Bessel Type Functions: Estimations, Series, Overconvergence

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Abstract. A family of the Bessel–Maitland functions are considered in this paper and some useful estimations are obtained for them. Series defined by means of these functions are considered and their behaviour on the boundaries of the convergence domains is discussed. Using the obtained estimations, necessary and sufficient conditions for the series overconvergence, as well as Hadamard type theorem are proposed.

INTRODUCTION

Let

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)}, \quad z,\ \nu \in \mathbb{C} \text{ and } \mu > -1$$

denote the generalization of the Bessel functions, introduced by Sir Edward Maitland Wright and known in the literature as Bessel–Maitland functions (named after his second name). Wright first introduced them for $\mu > 0$ and on a later stage he extended their definition for $\mu > -1$. We consider the family of functions $J_n^\mu(z)$ with nonnegative integer indices n, i. e.

$$J_n^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(n+\mu k+1)}, \quad z \in \mathbb{C},\ n = 0,1,2,\dots \text{ and } \mu > 0. \tag{1}$$

In the complex plane we consider series in these functions, that is, the series of the kind

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z) \qquad (\widetilde{J}_n^\mu(z) = n! z^n J_n^\mu(z)), \qquad a_n \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{2}$$

In a number of recently published papers (see e.g. [1] and also the book [2]), we have provided various results concerning the geometry of convergence of such a type of series. They are related to the domains where these series converge and where they do not, and moreover, where the convergence is uniform and where it is not. For the results discussed above, we need some suitable asymptotic formulae for the Bessel–Maitland functions for 'large' values of the indices. For example, the following representations holds true:

$$J_n^\mu(z) = \frac{1}{n!}\left(1+\theta_n^\mu(z)\right), \quad \theta_n^\mu(z) \to 0 \ \text{ as } \ n \to \infty, \tag{3}$$

in the whole complex plane. The functions $\theta_n^\mu(z)$ are holomorphic for $z \in \mathbb{C}$ and moreover $\lim_{n\to\infty} \theta_n^\mu(z) = 0$ uniformly on each nonempty compact subset K of the plane $\mathbb{C}$. Considering explicitly $\theta_n^\mu(z)$, the result from (3), has been made

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sharper. Namely, as it is proved in [2, Ch. 6, pp. 102, 112], there exists a constant $C = C(K)$, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$, the following inequality holds

$$|\theta_n^\mu(z)| \leq C/n^\mu. \tag{4}$$

The uniform convergence of $\theta_n^\mu(z)$ on the compact subsets of $\mathbb{C}$ follows from (4), as well.

PREVIOUS RESULTS

In this section we give some results for series of the kind (2) which refer to its domain of convergence and its behaviour 'near' the periphery of this domain and on its boundary as well. Such type of results are also obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials by Rusev, and resp. by the author (see e.g. [3] and [2]) for systems of some other special functions of fractional calculus, which are fractional-indices and multi-indices analogues of the Bessel functions and of the Mittag-Leffler functions (in the sense of [4] and [5]).

In what follows we use the notations $D(0;R)$ and $C(0;R)$ for the open disk centered at the origin with radius R and respectively for its boundary, i.e.

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\},$$

and $\widetilde{D}(0;R)$ for the circular domain

$$\widetilde{D}(0;R) = \{z : |z| > R,\ z \in \mathbb{C}\}.$$

Beginning with the domain of convergence, we note that the series (2) is absolutely convergent in the open disk $D(0;R)$ with a radius R, given by

$$R = \left[\limsup_{n\to\infty} (|a_n|)^{1/n}\right]^{-1}, \tag{5}$$

and divergent in the circular domain $\widetilde{D}(0;R)$. In other terms, the domain of convergence of the series (2) is the open disk $D(0;R)$ with a radius of convergence (5). Moreover, if this series converges at the point $z_0 \neq 0$, then it is absolutely convergent in the disk $D(0;|z_0|)$. Inside the open disk $D(0;R)$, i.e. on each closed disk $|z| \leq r$ $(r < R)$, the convergence is uniform. The very disk of convergence is not obligatorily a domain of uniform convergence and the series may even be divergent on its boundary.

Further if $F(z)$ is the sum of the series (2) in its domain of convergence $D(0;R)$ with $0 < R < \infty$ and if this series converges at the point z_0 of the boundary of $D(0;R)$, then the relation $\lim\limits_{z\to z_0} F(z) = \sum\limits_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0)$ holds true in a suitable angular domain g_{z_0} with a vertex at the point $z = z_0$. Near the vertex (in a special curve linear triangular part of g_{z_0}), the series (2) is uniformly convergent. The results discussed above are analogues of the classical Cauchy-Hadamard and Abel theorems and their proofs can be found in [2]). In general, there is no relation between the convergence (divergence) of the series (2) at points on the boundary of its disk of convergence and the regularity (singularity) of its sum at such points. But under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exist. Namely, if the coefficients of the series (2) with the unit disk of convergence tend to the zero, i.e. $\lim\limits_{n\to\infty} a_n = 0$, then the series converges, even uniformly, on each arc of the unit circle, all points of which (including the ends of the arc) are regular for the sum of the series (for details and proof, see [1, 2]). Just discussed result is known as Fatou type theorem and it is analogical to the corresponding classical theorem. Propositions referring to this property have also been established for series in the Laguerre and Hermite polynomials by Rusev, as well as in Mittag-Leffler systems [2].

Recall that, as it is well known, it is possible a given power series with a finite radius of convergence $0 < R < \infty$ to be convergent or divergent at some points of the boundary $C(0;R)$. These points could be regular or singular for its sum f, but the series diverges outside the domain of convergence. However, sometimes it is possible a subsequence of its partial sums to exist that converges in a neighbouhood of a regular point of the sum. In order to introduce the next two definitions ([6, Vol. 2, p. 500]) and to expose the results, obtained in this section, we first set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R), \qquad s_p(z) = \sum_{n=0}^{p} a_n z^n, \quad S_p(z) = \sum_{k=0}^{p} a_n \widetilde{J}_n^\mu(z), \qquad p = 0, 1, 2, \ldots \tag{6}$$

Definition 1 *A power series with a finite radius of convergence R is said to be overconvergent, if there exist a subsequence* $\left\{s_{p_k}\right\}_{k=1}^{\infty}$ *of the partial-sums sequence* $\left\{s_p\right\}_{p=0}^{\infty}$ *and a region G, containing the open disk* $D(0;R)$*,* $G \cap \partial D(0;R) \neq \varnothing$*, such that* $\left\{s_{p_k}\right\}$ *is uniformly convergent inside G.*

Definition 2 *We say that the function f (or the series), given by (6) possesses Hadamard gaps, if there exist two sequences* $\{p_n\}_{n=1}^{\infty}$ *and* $\{q_n\}_{n=1}^{\infty}$*, having the properties*

$$q_{n-1} \leq p_n < q_n, \quad \liminf_{n\to\infty} \frac{q_n}{p_n} = (1+\theta) \quad (\theta > 0),$$

and

$$a_k = 0 \quad for \quad p_n < k < q_n \ (n = 1, 2, \ldots).$$

Thus, beginning with the domain of convergence and series behaviour near its boundary, passing through the possible uniform convergence on an arbitrary closed arc of the boundary, all the points of which are regular for its sum f, we come to the natural question: '*What type of conditions should be imposed on the power series that ensure the existence of a subsequence* $\left\{s_{p_k}\right\}$*, convergent outside the disk of convergence?*'. The answer to this question is given in the early 20th century by Ostrowski. Namely, his classical result [7], see also [8], states that a given power series with Hadamard gaps and with existing regular points on the boundary of convergence disk is overconvergent. We draw the attention to the fact that merely the existence of Hadamard gaps does not imply overconvergence. For example, the power series $\sum_{n=0}^{\infty} a_{k_n} z^{k_n}$ with $k_{n+1} \geq (1+\theta)k_n$ $(\theta > 0)$ and $\limsup_{n\to\infty} (\,|a_{k_n}|\,)^{1/k_n} = 1$ possesses Hadamard gaps but nevertheless it is not overconvergent. Its natural boundary of analyticity is the unit circle $|z| = 1$ and that is nothing but the theorem about the gaps, belonging to Hadamard [9].

However, the occurrence of gaps is by no means necessarily connected with the non-continuability of a power series. The matter is quite different with the Ostrowski result, cited above. It turns out that the gap condition of this theorem is essentially connected with the problem. It is clear that if an arbitrary power series with radius of convergence grater than 1 is added to the overconvergent series with radius of convergence 1, then the new power series is also overconvergent. Even more, all the overconvergent power series are obtained in this manner. In other terms, the famous statement given by Ostrowski in [10] and [11] holds true. Namely, this result states that if a sequence $s_{n_k}(z)$ of partial sums of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with radius of convergence 1, is uniformly convergent in a neighbourhood of a point on the unit circle, then the representation $f(z) = h(z) + g(z)$ exists, where the power series $h(z)$ has Hadamard gaps and the radius of convergence of the power series $g(z)$ is greater than 1. A proposition of this kind, but for Fourier series, is established by Kovacheva in the paper [12], about the case of generalized Mittag-Leffler series see the recently published paper [13].

Remark 1 *To introduce the corresponding notions 'overconvergence' and 'gaps' for the series (2), the expression* z^n *has to be replaced by* $\widetilde{J}_n^{\mu}(z)$ *and, respectively, the sequence* $\{s_{p_k}\}$ *by the sequence* $\{S_{p_k}\}$*.*

First, we intend to formulate two statements, referring to the overconvergence of the considered series (2) (the details and proofs can be seen in [14] and also in [2]), as follows.

Theorem 1 (of overconvergence) *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers satisfying the condition* $\limsup_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$*,* $F(z)$ *be the sum of the series (2) in the unit disk* $D(0;1)$*,* $F(z)$ *have at least one regular point, belonging to the circle* $C(0;1)$*, and let* $F(z)$ *possess Hadamard gaps. Then the series (2) is overconvergent.*

Theorem 2 (of Hadamard type about the gaps) *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence of complex numbers satisfying the conditions* $\limsup_{n\to\infty} (\,|a_{k_n}|\,)^{1/k_n} = 1$ *for* $k_{n+1} \geq (1+\theta)k_n$ $(\theta > 0)$*,* $a_k = 0$ *for* $k_n < k < k_{n+1}$*, and* $F(z)$ *be the sum of the series (2) in the unit disk* $D(0;1)$*, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_{k_n} \widetilde{J}_{k_n}^{\mu}(z) \quad for \quad z \in D(0;1).$$

Then all the points of the unit circle $C(0;1)$ are singular for the function F, i.e. the unit circle is a natural boundary of analyticity for the series $\sum_{n=0}^{\infty} a_{k_n} \widetilde{J}^{\mu}_{k_n}(z)$.

AUXILIARY STATEMENTS: SOME USEFUL ESTIMATES

Let $\widetilde{\gamma} \subset \widetilde{D}(0;1)$ be a given circle and let the domain G be bounded by the circles $\widetilde{\gamma}$ and $C(0;1)$, i.e. G is their outside. Besides, let $R > 1$ and $C(0;R) \subset G$. Denoting

$$S_{k_n}(z) = \sum_{m=0}^{k_n} a_m \widetilde{J}^{\mu}_m(z), \qquad n = 1, 2, \ldots, \tag{7}$$

and

$$I_m = \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{S_{k_n}(z)}{z^{m+1}}\, dz, \tag{8}$$

the following result can be formulated.

Theorem 3 *If the series (2) has a radius of convergence 1 and the subsequence of its partial sums (7) converges uniformly on the curve $\widetilde{\gamma}$, then a positive number η exists, such that the following inequalities hold true*

$$\left|S_{k_n}(z)\right| \leq (Re^{-\eta})^{k_n} \tag{9}$$

for $z \in C(0;R)$ and

$$|I_m| \leq \frac{(Re^{-\eta})^{k_n}}{R^m}, \quad \textit{for} \quad m \leq k_n. \tag{10}$$

Proof Consider the function

$$u_n(z) = \frac{1}{k_n} \log \left| \frac{S_{k_n}(z)}{z^{k_n}} \right| = \frac{1}{k_n} \log \left|S_{k_n}(z)\right| - \log|z|, \quad n = 1, 2, \ldots, \tag{11}$$

which is subharmonic in the set, considered further.

Obviously, there exists a constant N_1 such that the inequality $u_n(z) < -q$ holds for $z \in \widetilde{\gamma}$ and $n > N_1$, with q not depending on z and n, and $q > 0$.

Now, letting $\varepsilon > 0$, we set $\delta = e^{-\varepsilon/2}$, whence $\delta < 1$. Since the maximum of the module $\left|\frac{S_{k_n}(z)}{z^{k_n}}\right|$ in the closure $[\widetilde{D}(0;\delta)]$ of the set $\widetilde{D}(0;\delta)$ is attained on the circle $C(0;\delta)$, then setting

$$m_{\delta,n} = \max_{|z|=\delta} \left|S_{k_n}(z)\right|,$$

the following inequality holds true in the set $|z| \geq \delta$

$$u_n(z) \leq \frac{1}{k_n} \log m_{\delta,n} - \log\delta = \frac{1}{k_n} \log m_{\delta,n} + \frac{\varepsilon}{2},$$

and therefore on the set $|z| \geq 1$ as well. Since $\lim\limits_{n\to\infty} \frac{1}{k_n} \log m_{\delta,n} = 0$, there exists a number N_2 such that for all the values of $n > N_2$, for $|z| > \delta$ and also in particular for $|z| \geq 1$, the inequality $u_n(z) \leq \varepsilon$ holds. Along with both circles $\widetilde{\gamma}$ and $C(0;1)$, we also consider the circle $C(0;R)$ with a radius $R > 1$ and lying in the domain G, bounded by the curves $\widetilde{\gamma}$ and $C(0;1)$. Now, letting $n > N = N(\varepsilon) = \max(N_1, N_2)$, we can write $u_n(z) < -q$ for $z \in \widetilde{\gamma}$ and $u_n(z) \leq \varepsilon$ for $|z| = 1$. As a result, the following estimation is obtained $u_n(z) \leq -\lambda_1 q + \lambda_2 \varepsilon$ for $z \in C(0;R)$ and $n > N$, with $\lambda_{1,2} > 0$ and $\lambda_{1,2}$ only depending on R (in fact, λ_1 is the minimum of a harmonic measure on the circle $C(0;R)$ [15, Ch. VIII, §4, the proof of Theorem 2]). Taking ε such that $-\lambda_1 q + \lambda_2 \varepsilon = -\eta < 0$ we conclude that $u_n(z) \le -\eta$ for $n > N$ and $|z| = R$, which produces the evaluation

$$\left|S_{k_n}(z)\right| \le \left(Re^{-\eta}\right)^{k_n}.$$

Now, using the latest inequality for $|S_{k_n}(z)|$, the following estimation is obtained for the modules $|I_m|$ of the integral (8)

$$|I_m| \le \frac{1}{2\pi}\frac{1}{R^{m+1}}\left(Re^{-\eta}\right)^{k_n} 2\pi R = \frac{\left(Re^{-\eta}\right)^{k_n}}{R^m}, \quad \text{for} \quad m \le k_n,$$

that is the desired inequality. □

In conclusion, note that if $\mu = 1$, the functions $\widetilde{J}_m^{\mu}$ coincide with the corresponding functions $\widetilde{J}_m$, and then the results obtained here refer to the series in the Bessel functions and its partial sums subsequence as well (see [16]).

Return to the integral I_m. We need it for evaluating the absolute value $|a_m|$. For this purpose, we first recall that $I_m = \operatorname{Res}_0\left(S_{k_n(z)}/z^{m+1}\right)$ and performing directly the integration in I_m for $m = 0, 1, \ldots, k_n$, we obtain consequently

$$I_0 = a_0, \quad I_1 = a_1 - a_0\,\frac{\Gamma(1)}{1!\Gamma(\mu+1)}, \quad I_2 = a_2 - a_1\,\frac{\Gamma(2)}{1!\Gamma(\mu+2)} + a_0\,\frac{\Gamma(1)}{2!\Gamma(2\mu+1)}, \tag{12}$$

$$I_3 = a_3 - a_2\,\frac{\Gamma(3)}{1!\Gamma(\mu+3)} + a_1\,\frac{\Gamma(2)}{2!\Gamma(2\mu+2)} - a_0\,\frac{\Gamma(1)}{3!\Gamma(3\mu+1)}.$$

In general,

$$I_m = a_m - a_{m-1}\,\frac{\Gamma(m)}{1!\Gamma(\mu+m)} + a_{m-2}\,\frac{\Gamma(m-1)}{2!\Gamma(2\mu+m-1)} + \ \ldots \tag{13}$$

$$+a_2\,\frac{(-1)^{m-2}\Gamma(3)}{(m-2)!\Gamma((m-2)\mu+3)} + a_1\,\frac{\Gamma(2)(-1)^{m-1}}{(m-1)!\Gamma((m-1)\mu+2)} + a_0\,\frac{(-1)^m\Gamma(1)}{m!\Gamma(m\mu+1)}.$$

Further, denoting

$$\alpha_{s,m} = \frac{\Gamma(s+1)}{(m-s)!\,\Gamma((m-s)\mu+s+1)} \quad (0 \le s \le m), \tag{14}$$

the relation (13) takes the form

$$I_m = a_m - a_{m-1}\,\alpha_{m-1,m} + a_{m-2}\,\alpha_{m-2,m} + \ \ldots + (-1)^{m-2}\,a_2\,\alpha_{2,m} + (-1)^{m-1}\,a_1\,\alpha_{1,m} + (-1)^m\,a_0\,\alpha_{0,m}\,. \tag{15}$$

As a matter of fact, the following helping result can be formulated under additional condition for the parameter μ.

Lemma 1 *If $\mu \ge 1$ and $0 \le s \le m$, the following inequalities hold true*

$$\Gamma((m-s)\mu+s+1) \ge \Gamma(m+1) = m!, \quad 0 < \alpha_{s,m} \le \frac{s!}{(m-s)!\,m!}. \tag{16}$$

Proof The truthfulness of (16) immediately follows keeping in view that the argument $(m-s)\mu + s + 1 \ge m + 1 \ge 2$ and $\Gamma-$function increases in this case. □

Theorem 4 *If the parameter $\mu \ge 1$, the series (2) has a radius of convergence 1, the subsequence (7) of its partial sums converges uniformly on the curve $\widetilde{\gamma}$, and the radius R has the property $1 < R < \sqrt{2}$, then the coefficients a_m in (7) satisfy the inequalities*

$$|a_0| \le \left(Re^{-\eta}\right)^{k_n}, \quad |a_1| \le \frac{5}{2R}\left(Re^{-\eta}\right)^{k_n}, \tag{17}$$

$$|a_2| \le \frac{7}{2R^2}\left(Re^{-\eta}\right)^{k_n}, \quad |a_3| \le \frac{7}{2R^3}\left(Re^{-\eta}\right)^{k_n}, \tag{18}$$

and

$$|a_m| \le \frac{3}{R^m}\left(Re^{-\eta}\right)^{k_n}, \quad \textit{for} \quad 4 \le m \le k_n. \tag{19}$$

Proof Using (10), (12), and Lemma 1, the first estimation (17), concerning $|a_0|$, automatically follow, namely

$$|a_0| = |I_0| \leq \left(Re^{-\eta}\right)^{k_n} .$$

Respectively, for $|a_1|$ the next inequality holds

$$|a_1| \leq |I_1| + |a_0| \leq \frac{1}{R}\left(Re^{-\eta}\right)^{k_n} + \left(Re^{-\eta}\right)^{k_n} \leq \frac{(1+\sqrt{2})}{R}\left(Re^{-\eta}\right)^{k_n} ,$$

whence (17) follows immediately. Analogically,

$$\begin{aligned}|a_2| &\leq |I_2| + |a_1|\frac{\Gamma(2)}{1!\Gamma(3)} + |a_0|\frac{\Gamma(1)}{2!\Gamma(3)} = |I_2| + |a_1|\frac{1}{2!} + |a_0|\frac{1}{2!2!} \\ &\leq \left(\frac{1}{R^2} + \frac{5}{4R} + \frac{1}{4}\right)\left(Re^{-\eta}\right)^{k_n} \leq \left(\frac{1}{R^2} + \frac{15}{8R^2} + \frac{1}{2R^2}\right)\left(Re^{-\eta}\right)^{k_n} \\ &= \frac{27}{8R^2}\left(Re^{-\eta}\right)^{k_n} < \frac{7}{2R^2}\left(Re^{-\eta}\right)^{k_n} ,\end{aligned}$$

and

$$\begin{aligned}|a_3| &= |I_3| + |a_2|\frac{2!}{1!3!} + |a_1|\frac{1}{2!3!} + |a_0|\frac{1}{3!3!} \\ &\leq \left(\frac{1}{R^3} + \frac{4}{3R^2} + \frac{5}{3!4R} + \frac{1}{3!3!}\right)\left(Re^{-\eta}\right)^{k_n} \\ &\leq \left(\frac{1}{R^3} + \frac{2}{R^3} + \frac{5}{12R^3} + \frac{1}{12R^3}\right)\left(Re^{-\eta}\right)^{k_n} = \frac{7}{2R^3}\left(Re^{-\eta}\right)^{k_n} ,\end{aligned}$$

which verifies the validity of (18).

In order to prove (19), we first check its validity for $m = 4$, namely

$$\begin{aligned}|a_4| &\leq |I_4| + \frac{|a_3|}{4} + \frac{|a_2|}{4!} + \frac{|a_1|}{3!4!} + \frac{|a_0|}{4!4!} \\ &\leq \left(\frac{1}{R^4} + \frac{7}{8R^3} + \frac{7}{4!2R^2} + \frac{5}{3!4!2R} + \frac{1}{4!4!}\right)\left(Re^{-\eta}\right)^{k_n} \\ &\leq \left(\frac{1}{R^4} + \frac{21}{16R^4} + \frac{7}{24R^4} + \frac{5}{16R^4} + \frac{1}{144R^4}\right)\left(Re^{-\eta}\right)^{k_n} \\ &\leq \left(\frac{2}{R^4} + \frac{11}{12R^4} + \frac{1}{144R^4}\right)\left(Re^{-\eta}\right)^{k_n} < \frac{3}{R^4}\left(Re^{-\eta}\right)^{k_n} ,\end{aligned}$$

which confirms (19) for $m = 4$.

Assuming that

$$|a_m| \leq \frac{3}{R^m}\left(Re^{-\eta}\right)^{k_n} , \quad \text{for } 4 \leq m < k_n,$$

the next inequalities

$$\begin{aligned}|a_{m+1}| &\leq |I_{m+1}| + \frac{|a_m|}{m+1} + \frac{|a_{m-1}|}{2!m(m+1)} + \frac{|a_{m-2}|}{3!(m-1)m(m+1)} + \dots \\ &+ \frac{3!|a_3|}{(m-2)!(m+1)!} + \frac{2!|a_2|}{(m-1)!(m+1)!} + \frac{|a_1|}{m!(m+1)!} + \frac{|a_0|}{(m+1)!(m+1)!} \\ &\leq \left(\frac{1}{R^{m+1}} + \frac{1}{R^{m+1}} + \underbrace{\frac{1}{(m+1)R^{m+1}} + \dots + \frac{1}{(m+1)R^{m+1}}}_{m}\right)\left(Re^{-\eta}\right)^{k_n} \\ &\leq \frac{3}{R^{m+1}}\left(Re^{-\eta}\right)^{k_n} , \quad \text{for } 5 \leq m+1 \leq k_n,\end{aligned}$$

immediately can be written. That is why, (19) follows by induction. □

OVERCONVERGENCE RESULTS

Analogically to the power series case, the adding of a series of the kind (2) with radius of convergence greater than 1, to the overconvergent series (2) with radius of convergence 1, keeps the overconvergence. Further, the inverse proposition of Theorem 1 can be proposed, namely, the overconvergent series (2) can be represented as a sum of two series – the first with radius of convergence greater than 1 and the second with Hadamard gaps. More precisely, the following theorem holds true.

Theorem 5 *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup_{n\to\infty} (|a_n|)^{1/n} = 1$, the parameter $\mu \geq 1$, $F(z)$ be the sum of the series (2) in the unit disk $D(0; 1)$ and let the series (2) be overconvergent. Then $F(z)$ can be represented in the form*

$$F(z) = H(z) + G(z), \tag{20}$$

where the series $H(z)$ possesses Hadamard gaps and the radius of convergence of the series $G(z)$ is greater than 1.

Proof From the overconvergence of the series (2) it follows that there exist a circle $\widetilde{\gamma} \subset \widetilde{D}(0; 1)$ and a subsequence $\{S_{k_n}(z)\}$ of its partial sums, defined by (7), which is uniformly convergent on the curve $\widetilde{\gamma}$. We select the indices k_n in such a way that

$$k_{n+1} > \alpha_n k_n, \qquad n = 1, 2, \ldots \quad (\liminf_{n\to\infty} \alpha_n > 1). \tag{21}$$

Considering the domain G, bounded by both curves $C(0; 1)$ and $\widetilde{\gamma}$, we also consider the circle $C(0; R)$ with a radius $R > 1$ and lying in G. Without the loss of generality, the distance between the curves $C(0; 1)$ and $\widetilde{\gamma}$ can be assumed less than or equal to $\sqrt{2} - 1$. This implies that $R \leq \sqrt{2}$, because it holds true for the radius R of every possible circle $C(0; R) \subset G$.

In particular, setting $\alpha = \liminf_{n\to\infty} \alpha_n$ and taking $1/\alpha < p' < 1$, $p'k_n < m \leq k_n$, the inequality $R^{-m} < R^{-p'k_n}$ follows, which implies

$$|a_m| \leq 3 \left(\frac{Re^{-\eta}}{R^{p'}}\right)^{k_n}, \quad \text{for} \;\; p'k_n < m \leq k_n. \tag{22}$$

Choosing p' so close to the 1 that $\left(Re^{-\eta}R^{-p'}\right) = \rho < 1$, inequality (22) gets the form

$$|a_m| \leq 3\rho^{k_n} \leq 3\rho^m \quad (\rho < 1), \qquad \text{for} \;\; p'k_n < m \leq k_n. \tag{23}$$

Further, denoting $p_n = [p'k_n]$ (where $[p'k_n]$ means the entire part of $p'k_n$), we consider the grouped series

$$G(z) = \sum_{n=1}^{\infty} \widetilde{J}_{p_n+1,k_n}(z), \quad \widetilde{J}_{p_n+1,k_n}(z) = a_{p_n+1}\widetilde{J}^{\mu}_{p_n+1}(z) + \ldots + a_{k_n}\widetilde{J}^{\mu}_{k_n}(z). \tag{24}$$

Bearing in mind the condition (21) and also that $1/\alpha < p' < 1$, the following inequality can be produced for the subscripts in $\widetilde{J}_{p_n+1,k_n}$

$$p_n + 1 \leq k_n < \frac{1}{\alpha}k_{n+1} < p'k_{n+1} < [p'k_{n+1}] + 1 = p_{n+1} + 1. \tag{25}$$

On the one hand, the highest index in the summand $\widetilde{J}_{p_n+1,k_n}(z)$ is k_n and on the other hand, the lowest index of the next summand is $p_{n+1} + 1$. Hence, individual functions in different terms of (24) do not overlap. So that the development of all terms of (24) can be considered as in the functions $\widetilde{J}^{\mu}_m(z)$ with indices $m \in \{p_n + 1, \ldots, k_n\}_{n=1}^{\infty}$. Eventually, the inequality (23) shows that the functions series (24) considered as a series in the Bessel–Maitland functions $\widetilde{J}^{\mu}_m(z)$, converges in the disk $D(0; 1/\rho)$ with $1/\rho > 1$.

The additional series

$$H(z) = \sum_{m=0}^{\infty} \widetilde{a}_m \widetilde{J}^{\mu}_m(z), \qquad \widetilde{a}_m = 0 \;\; \text{for} \;\; m \in \{p_n + 1, \ldots, k_n\}_{n=1}^{\infty}, \quad \widetilde{a}_m = a_m \text{ otherwise}, \tag{26}$$

obviously is lacunary. In order to prove this, we first note that $\widetilde{a}_m = 0$ for $p_n < m < k_n + 1$, $n = 1, 2, \ldots$, and consider (26). Then, setting $\alpha_0 = \frac{1}{p'} > 1$ and $q_n = k_n + 1$, and bearing in mind that $p_n = [p'k_n] \leq p'k_n$, we obtain that

$$\alpha_0 p_n = \alpha_0[p'k_n] \leq \alpha_0 p'k_n = k_n < k_n + 1 = q_n \quad \text{and } \widetilde{a}_m = 0 \text{ for } p_n < m < q_n.$$

Therefore $\alpha_0 p_n < q_n$ with $\alpha_0 > 1$, which means that the series (26) has Hadamard gaps. □

In conclusion note that the particular case $\mu = 1$ leads to the results, obtained in [16].

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